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Zero sets of τ-functions and hidden hierarchies of KdV type

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Abstract. The zero sets of KdV τ-functions are characterized in terms of the stratification of the infinite Grassmannian. It is shown that these sets are related to integrable hierarchies arising from Schrödinger equations with energy-dependent potentials.

1. Introduction

The analysis of τ-functions in the framework of the infinite-dimensional Grassmannian Gr [1–3] has been relevant not only in the development of the theory of nonlinear integrable systems of Korteweg–de Vries (KdV) type but also in providing a bridge between these systems and important applications of quantum field theory to algebraic geometry, quantum gravity and string theory [4]. Integrable systems such as those of the Kadomtsev–Petviashvili (KP), KdV or Gelfand–Dikii hierarchies are described by flows in the so-called *big cell* of Gr. For example, in the case of $\mathrm{Gr}^{(2)}$, the part of Gr which is relevant for the KdV hierarchy, every element $W \in \mathrm{Gr}^{(2)}$ determines a flow $W(\boldsymbol{t})$ in $\mathrm{Gr}^{(2)}$ and a solution of the hierarchy of the form

$$u_W(\boldsymbol{t}) = -2\partial_1^2 \ln \tau_W(\boldsymbol{t}) \qquad \boldsymbol{t} := (t_1, t_3, t_5, \ldots)$$

where $\tau_W(\boldsymbol{t})$ is the τ-function associated to W. This solution is defined only for those $\boldsymbol{t}$ such that $\tau_W(\boldsymbol{t}) \neq 0$, or equivalently, provided $W(\boldsymbol{t})$ is in the big cell of Gr.

Despite the fact that the big cell is a dense open set of Gr, there are other sectors in $\mathrm{Gr}^{(2)}$ which deserve attention. Thus, $\mathrm{Gr}^{(2)}$ admits a partition into strata

$$\mathrm{Gr}^{(2)} = \bigcup_{m\geqslant 0} \Sigma_m$$

and only the stratum Σ_0 is in the big cell. In a recent work by Adler and van Moerbeke [5] it was shown that the strata different from the big cell of the general Grassmannian Gr are essential for describing the blow-up behaviours of the Baker functions of the KP hierarchy. Similar considerations for the N-periodic Toda flows can be found in [6].

The present paper shows that the strata $\Sigma_m, m \geqslant 1$ of $\mathrm{Gr}^{(2)}$, support the flows of the integrable hierarchies associated to Schrödinger equations with energy-dependent potentials

$$\partial_x^2 f = \left(\lambda^{2m+1} + \sum_{n=0}^{2m} \lambda^n u_n(x)\right) f \qquad \lambda := k^2.$$

These hierarchies were introduced in [7, 8] and further generalized and studied in ([9–12]). In what follows they will be referred to as the hidden $(2m+1)$th KdV hierarchies ($\mathrm{hKdV}_{(2m+1)}$) since their flows take place outside the big cell.

The majority of our analysis is concerned with the close link it establishes between the hKdV hierarchies and the zero sets of KdV τ-functions in the infinite-dimensional space $\mathbb{C}^\infty = \{\boldsymbol{t} = (t_1, t_3, t_5, \ldots), t_i \in \mathbb{C}\}$. Thus, it is proved that, as a function of t_1, $\tau_W(\boldsymbol{t})$ can have zeros of orders $\ell_m := m(m+1)/2$ ($m \geqslant 1$) only, and that the set of ℓ_m-order zeros of τ_W is characterized by some associated solutions of the $\mathrm{hKdV}_{(2m+1)}$ hierarchy. As a consequence, a method is provided for characterizing solutions of the hKdV hierarchies from τ-functions of the standard KdV hierarchy.

2. Zeros of τ-functions and the stratification of the Grassmannian

Let H be the Hilbert space of all square-integrable functions on the unit circle S^1 of the complex plane. It can be decomposed as the direct sum $H = H_+ \oplus H_-$ of the closed subspaces H_+ and H_- spanned by the basis elements $\{k^n\}$ with $n \geqslant 0$ and $n < 0$, respectively. We will consider the Grassmannian, Gr, of all subspaces W of H such that:

(i) the orthogonal projections $P_\pm : W \longrightarrow H_\pm$ are operators of Fredholm and compact types, respectively;

(ii) the virtual dimension of W (i.e. the index of P_+) is zero.

It can be proved that Gr constitutes a connected Banach manifold which exhibits a stratified structure [13, 1]. To describe this structure let us introduce the set $\boldsymbol{S}_0$ of increasing sequences of integers

$$S = \{s_0, s_1, s_2, \ldots\}$$

such that $s_n = n$ for all sufficiently large n. We may associate to each $W \in \mathrm{Gr}$ the set of integers

$$S_W = \{n \in \mathbb{Z} : \exists w \in W \text{ of order } n\}.$$

(An element $w \in H$ is said to be of finite order n if it can be expressed in the form $w = \sum_{m \leqslant n} a_m k^m$, with $a_n \neq 0$.) As a consequence of the fact that the virtual dimension of W is zero, it follows that $S_W \in \boldsymbol{S}_0$. Thus, given $S \in \boldsymbol{S}_0$ we may define the subset of Gr

$$\Sigma_S = \{W \in \mathrm{Gr} : S_W = S\}$$

which is called the stratum associated with S. In any $W \in \mathrm{Gr}$ the elements of finite order form a dense open subspace denoted by W^{alg}. Therefore, W belongs to Σ_S when W^{alg} has a basis $\{w_n : n \geqslant 0\}$ with w_n of order s_n.

The stratum Σ_S is a submanifold of Gr of finite codimension given by

$$\ell(S) := \mathrm{codim}\Sigma_S = \sum_{n \geqslant 0} (n - s_n).$$

In particular, if S is the set of non-negative integers the corresponding stratum has codimension zero and constitutes a dense open subset of Gr which is called *the big cell*.

In the analysis of the KdV hierarchy one is lead to consider the subset of Gr given by

$$\mathrm{Gr}^{(2)} = \{W \in \mathrm{Gr} : k^2 W \subset W\}.$$

Here k^2 denotes the action of the multiplication operator by the function k^2. It is obvious that $S_W + 2 \subset S_W$ for all $W \in \mathrm{Gr}^{(2)}$, and as a consequence the stratification of $\mathrm{Gr}^{(2)}$ turns out to be

$$\mathrm{Gr}^{(2)} = \bigcup_{m \geqslant 0} \Sigma_m \qquad \Sigma_m := \Sigma_{S_m} \cap \mathrm{Gr}^{(2)} \tag{1}$$

where

$$S_m = \{-m, -m+2, -m+4, \ldots, m, m+1, m+2, \ldots\}. \tag{2}$$

Let us now consider the group Γ_+ of holomorphic maps

$$g : D_0 \longrightarrow \mathbb{C}^\times \qquad D_0 = \{k \in \mathbb{C} : |k| \leqslant 1\}.$$

This group acts by multiplication operators on $\mathrm{Gr}^{(2)}$, so that given $W \in \mathrm{Gr}^{(2)}$ then for appropriate $\boldsymbol{t} := (t_1, t_3, t_5, \ldots) \in \mathbb{C}^\infty$ we may define

$$W(\boldsymbol{t}) := \{f_0(k, \boldsymbol{t})^{-1} w : w \in W\} \tag{3}$$

where

$$f_0(k, \boldsymbol{t}) = \exp\left(\sum_{n \geqslant 0} t_{2n+1} k^{2n+1}\right). \tag{4}$$

As is proved in [1] $W(\boldsymbol{t})$ belongs to the big cell Σ_0 for *almost all* $\boldsymbol{t}$. This is so because there exists a non-zero holomorphic function $\tau_W(\boldsymbol{t})$ associated with W such that the Baker function defined by

$$f(k, \boldsymbol{t}) = f_0(k, \boldsymbol{t}) \frac{\tau_W(\boldsymbol{t} - \boldsymbol{\varepsilon}(k))}{\tau_W(\boldsymbol{t})} \tag{5}$$

with

$$\boldsymbol{\varepsilon}(k) = \left(\frac{1}{k}, \frac{1}{3k^3}, \ldots, \frac{1}{(2n+1)k^{2n+1}}, \ldots\right)$$

belongs to W for all $\boldsymbol{t}$ such that $\tau_W(\boldsymbol{t}) \neq 0$. In this way, and taking into account that the derivatives of f with respect to the variables t_n are also members of W, it is trivial to prove that provided $\tau_W(\boldsymbol{t}) \neq 0$ the subspace $W(\boldsymbol{t})$ contains elements w_n of order n for all $n \geqslant 0$:

$$w_n(k, \boldsymbol{t}) := f_0(k, \boldsymbol{t})^{-1} \partial_x^n f(k, \boldsymbol{t}) \qquad x := t_1.$$

Therefore, $W(\boldsymbol{t}) \in \Sigma_0$ and, as a consequence, there exist decompositions of the form

$$\begin{aligned} \partial_x^2 f &= (\lambda + u_W(\boldsymbol{t})) f \qquad \lambda := k^2 \\ \partial_{2n+1} f &= a(k^2, \boldsymbol{t}) f + b(k^2, \boldsymbol{t}) \partial_x f \end{aligned}$$

where

$$\partial_{2n+1} := \frac{\partial}{\partial t_{2n+1}} \qquad n \geqslant 0$$

and a and b are polynomials in k^2. By imposing the compatibility between these equations one obtains the standard KdV hierarchy of evolution equations for the function

$$u_W(\boldsymbol{t}) := -2\partial_1^2 \ln \tau_W(\boldsymbol{t}).$$

Now we consider one of the main points of our discussion: the analysis of the zero set of $\tau_W(\boldsymbol{t})$, or equivalently, the set of singularities of the Baker function $f(k, \boldsymbol{t})$ and the corresponding solution $u_W(\boldsymbol{t})$ of the KdV hierarchy. A detailed study of the zero sets of the τ-functions for the general Grassmannian Gr has been provided in [5], where a method is given for desingularizing the Baker functions by means of Bäcklund transformations. In our present study we only require some specific properties of the τ-functions for the reduced Grassmannian $\mathrm{Gr}^{(2)}$, which are included in the next theorem and its corollary. More general estimates for the behaviour of the τ-functions near the zeros are given in theorem 7.3 of [5].

Theorem 1. Let $\boldsymbol{t}_0 = (t_{01}, t_{03}, t_{05}, \ldots)$ be a zero of $\tau_W(\boldsymbol{t})$. Then there exists an integer $m > 0$ such that the function $\tau_W(t_1, t_{03}, t_{05}, \ldots)$ has a zero of order

$$\ell_m := \frac{m(m+1)}{2}$$

at $t_1 = t_{01}$.

Proof. Since $W \in \mathrm{Gr}^{(2)}$ then it follows that $W(\boldsymbol{t}_0) \in \mathrm{Gr}^{(2)}$, so that according to equation (1) and taking into account that $\tau_W(\boldsymbol{t}_0) = 0$, we deduce that $W(\boldsymbol{t}_0)$ is in one of the strata Σ_m for some $m > 0$. Now, from proposition 8.6 of [1] we have that for any $V \in \mathrm{Gr}$

$$\tau_V(t_1, 0, 0, \ldots) = ct_1^{\ell} + \mathcal{O}(t_1^{\ell+1})$$

where $c \neq 0$ and ℓ is the codimension of the stratum of Gr containing V. Moreover, it is easy to find that

$$\operatorname{codim} \Sigma_{S_m} =: \ell(\Sigma_m) = \ell_m$$

and therefore

$$\tau_{W(\boldsymbol{t}_0)}(x, 0, 0, \ldots) = cx^{\ell_m} + \mathcal{O}(x^{\ell_m+1})$$

with $c \neq 0$. Furthermore, according to the following property of τ-functions—which derives from equation (3.4) of [1]

$$\tau_V(\boldsymbol{t} + \boldsymbol{t}') = \tau_{V(\boldsymbol{t})}(\boldsymbol{t}')$$

we have

$$\tau_W(t_{01} + x, t_{03}, t_{05}, \ldots) = \tau_{W(\boldsymbol{t}_0)}(x, 0, 0, \ldots).$$

Hence the statement of the theorem immediately follows. □

As a consequence of this result we see that the minimal order ℓ_m for which a derivative of the form $\partial_1^{\ell_n} \tau_W(\boldsymbol{t}_0)$ does not vanish characterizes the stratum Σ_m containing $W(\boldsymbol{t}_0)$. Thus, we may state the following.

Corollary 1. The following statements are equivalent:

(1) $W(\boldsymbol{t}_0)$ is in the stratum Σ_m.

(2) The τ-function of W satisfies

$$\partial_1^n \tau_W(\boldsymbol{t}_0) = 0 \qquad 0 \leqslant n < \ell_m \qquad \partial_1^{\ell_m} \tau_W(\boldsymbol{t}_0) \neq 0. \tag{6}$$

(3) The τ-function of W satisfies

$$\partial_1^{\ell_n} \tau_W(\boldsymbol{t}_0) = 0 \qquad 0 \leqslant n < m \qquad \partial_1^{\ell_m} \tau_W(\boldsymbol{t}_0) \neq 0. \tag{7}$$

3. Zeros of τ-functions and hKdV hierarchies

We are now in a position to analyse the relationship between the zero sets of τ-functions and the hKdV hierarchies. Let us suppose given $W \in \mathrm{Gr}^{(2)}$ and let us denote by Z_W the zero set of the corresponding τ-function τ_W. According to theorem 1 there is a partition of Z_W of the form

$$Z_W = \bigcup_{m \geqslant 1} Z_W^{\ell_m}$$

where $Z_W^{\ell_m}$ stands for the set of zeros $\boldsymbol{t}_0 = (t_{01}, t_{03}, t_{05}, \ldots)$ of $\tau_W(\boldsymbol{t})$ such that the function $\tau_W(t_1, t_{03}, t_{05}, \ldots)$ has a zero of order ℓ_m at $t_1 = t_{01}$. From corollary 1 we see that $Z_W^{\ell_m}$ can be characterized as the set of solutions $\boldsymbol{t} \in \mathbb{C}^\infty$ of the system of m equations

$$\partial_1^{\ell_n} \tau_W(\boldsymbol{t}) = 0 \qquad 0 \leqslant n < m \tag{8}$$

satisfying $\partial_1^{\ell_m} \tau_W(\boldsymbol{t}) \neq 0$.

The set Z_W is an *analytic set* in $\mathbb{C}^\infty$ [14], so that it can be considered as a union of complex manifolds. Suppose we are able to find a *patch* in Z_W described by a mapping $\mathbb{D} \subset \mathbb{C}^\infty \longrightarrow \mathbb{Z}_{\mathbb{W}}^{\ell_>}$ of the following form

$$\boldsymbol{t}_m := (t_{2m+1}, t_{2m+3}, t_{2m+5}, \ldots) \longmapsto \boldsymbol{t}(\boldsymbol{t}_m) := (b_1(\boldsymbol{t}_{(m)}), \ldots, b_m(\boldsymbol{t}_m), \boldsymbol{t}_m) \quad (9)$$

where the functions b_i are m complex-valued functions depending on $\boldsymbol{t}_m$. This means that $\boldsymbol{t}(\boldsymbol{t}_m)$ is required to satisfy equation (8) and $\partial_1^{\ell_m}\tau_W(\boldsymbol{t}(\boldsymbol{t}_m)) \neq 0$ for all $\boldsymbol{t}_m \in \mathbb{D}$. Notice that the functions b_i can be found by solving (8) with respect to the first m variables t_{2i+1}.

We are going to see that patches $\boldsymbol{t}(\boldsymbol{t}_m)$ are associated with solutions of the $\mathrm{hKdV}_{(2m+1)}$ hierarchy. From corollary 1 we have that $W(\boldsymbol{t}(\boldsymbol{t}_m)) \in \Sigma_m$ for all $\boldsymbol{t}_m \in \mathbb{D}$, and therefore there exists a unique function in $W(\boldsymbol{t}(\boldsymbol{t}_m))$ of order $-m$

$$\hat{f}(k, \boldsymbol{t}_m) = \frac{1}{k^m}\left(1 + \frac{a_1(\boldsymbol{t}_m)}{k} + \cdots + \frac{a_n(\boldsymbol{t}_m)}{k^n} + \cdots\right). \quad (10)$$

Theorem 2. The function

$$f(k, \boldsymbol{t}_m) = f_0(k, \boldsymbol{t}(\boldsymbol{t}_m))\hat{f}(k, \boldsymbol{t}_m) \quad (11)$$

satisfies the Schrödinger equation with an energy-dependent potential

$$\partial_x^2 f = \left(\lambda^{2m+1} + \sum_{n=0}^{2m} \lambda^n u_n(\boldsymbol{t}_m)\right) f \qquad x := t_{2m+1} \qquad \lambda := k^2 \quad (12)$$

and a system of equations of the form

$$\partial_{2n+1} f = a(k^2, \boldsymbol{t}_m) f + b(k^2, \boldsymbol{t}_m)\partial_x f \qquad n \geqslant m+1 \quad (13)$$

where a and b are polynomials in k^2.

Proof. From expansion (10) we have that for all $n \geqslant 0$ the functions

$$k^{2n} f_0^{-1} f \qquad k^{2n} f_0^{-1}\partial_x f \quad (14)$$

are elements of $W(\boldsymbol{t}(\boldsymbol{t}_m))$ of orders $2n-m$ and $2n+m+1$, respectively. Hence, due to the fact that $W(\boldsymbol{t}(\boldsymbol{t}_m)) \in \Sigma_m$, it follows that the functions in (14) form a basis of $W(\boldsymbol{t}(\boldsymbol{t}_m))^{\mathrm{alg}}$.

Moreover, by denoting

$$b(\boldsymbol{t}_m) := \sum_{n=1}^{m} k^{2n-1} b_n(\boldsymbol{t}_m)$$

and

$$g := \partial_x^2 f - (\partial_x b + k^{2m+1})^2 f - (2k^{2m}\partial_x a_1) f$$

it is obvious that $g \in W$. Furthermore,

$$f_0^{-1} g = \mathcal{O}(k^{m-1}) \quad (15)$$

so that $f_0^{-1} g$ belongs to $W(\boldsymbol{t}(\boldsymbol{t}_m))$ and has an order not greater than $m-1$. Hence, there exists a decomposition

$$f_0^{-1} g = \sum_{n=0}^{m-1} k^{2n} c_n(\boldsymbol{t}_m) f_0^{-1} f$$

and this implies an equation of the type (12).

In a similar way one proves that (11) satisfies equations of the form (13). □

The compatibility conditions for (12) and (13) lead [10] to the hKdV hierarchies associated with the Schrödinger spectral problem (12). For example, if we take $m = 1$ and $t = t_5$ the coefficients of the potential function satisfy the evolution equations

$$\begin{aligned}\partial_t u_0 &= \tfrac{1}{4}\partial_x^3 u_2 - u_0\partial_x u_2 - \tfrac{1}{2}u_2\partial_x u_0\\ \partial_t u_1 &= -\tfrac{1}{2}u_2\partial_x u_1 - u_1\partial_x u_2 + \partial_x u_0\\ \partial_t u_2 &= -\tfrac{3}{2}u_2\partial_x u_2 + \partial_x u_1\end{aligned} \tag{16}$$

where $t := t_5$ and $x := t_3$.

The above analysis provides a method for generating solutions to the hKdV hierarchies from elements $W \in \mathrm{Gr}^{(2)}$. The starting point is the τ-function $\tau_W(\boldsymbol{t})$ corresponding to W. Suppose that for a given $m \geqslant 1$ the system (8) can be solved with respect to $(t_1, \ldots, t_{2m-1})$ in terms of m functions of $\boldsymbol{t}_m = (t_{2m+1}, t_{2m+3}, \ldots)$

$$t_{2i-1} = b_i(\boldsymbol{t}_m) \qquad i = 1, \ldots, m.$$

Then, function (11) determines a wavefunction of the $\mathrm{hKdV}_{(2m+1)}$ hierarchy on the domain $\mathbb{D}$ of points $\boldsymbol{t}_m$ such that

$$\partial_1^{\ell_m}\tau_W(\boldsymbol{t}(\boldsymbol{t}_m)) \neq 0.$$

Solutions of the members of the hierarchy can be derived from the functions b_i and a_n arising in expansion (10). For example, if $m = 1$, we have

$$\begin{aligned}u_0 &= 2\partial_x a_3 - 2a_2\partial_x a_1 + a_1\partial_x^2 b_1 + 2\partial_x a_1\partial_x b_1\\ u_1 &= (\partial_x b_1)^2 + 2\partial_x a_1\\ u_2 &= 2\partial_x b_1.\end{aligned} \tag{17}$$

Our next theorem shows how to determine the explicit form of (10) from the τ-function of W.

Theorem 3. If $W(\boldsymbol{t}_0)$ is in the stratum Σ_m then the function $\hat{f}(k, \boldsymbol{t}_0)$ is given by

$$\hat{f}(k, \boldsymbol{t}_0) = \frac{\partial_1^{\ell_{m-1}}\tau_W(\boldsymbol{t}_0 - \boldsymbol{\varepsilon}(k))}{\partial_1^{\ell_{m-1}}P_m(\boldsymbol{\partial})\tau_W(\boldsymbol{t}_0)} \tag{18}$$

where $P_m(\boldsymbol{\partial})$, $\boldsymbol{\partial} := (\partial_1, \partial_3, \ldots)$, is obtained from the identity

$$\exp(-\boldsymbol{\varepsilon}(k)\cdot\boldsymbol{\partial}) = \sum_{n\geqslant 0}\frac{1}{k^n}P_n(\boldsymbol{\partial}).$$

Proof. The proof of this result is based on the properties of the decomposition of τ-functions in terms of Schur functions [1]

$$\tau_V(\boldsymbol{t}) = \sum_S w^S F_S(\boldsymbol{t}). \tag{19}$$

Each F_S is a polynomial in the t_i, homogeneous of weight $\ell(S)$ if we give t_i weight i, with $\ell(S)$ given by the codimension of the stratum Σ_S. It turns out [1] that the minimal weight of the terms in (19) is the codimension of the stratum on which V lies. Hence, by taking into account that

$$\tau_W\left(t_{01} - \frac{1}{k} + x, t_{03} - \frac{1}{3k^3}, t_{05} - \frac{1}{5k^5}, \ldots\right) = \tau_{W(\boldsymbol{t}_0)}\left(x - \frac{1}{k}, -\frac{1}{3k^3}, -\frac{1}{5k^5}, \ldots\right)$$

from the assumption $W(\boldsymbol{t}_0) \in \Sigma_m$ we deduce that

$$\partial_1^n\tau_W(\boldsymbol{t}_0 - \boldsymbol{\varepsilon}(k)) = \mathcal{O}\left(\frac{1}{k^{\ell_m - n}}\right). \tag{20}$$

Hence, the minimal order $n_{\min}$ for which $\partial_1^{n_{\min}}\tau_W(\boldsymbol{t}_0-\varepsilon(k))$, as a function of k, is not identically zero and must satisfy $n_{\min}\geqslant \ell_{m-1}$. Otherwise $W(\boldsymbol{t}_0)$ would admit elements of degree $d<m$ and this would contradict the assumption $W(\boldsymbol{t}_0)\in\Sigma_m$. Let us see that $n_{\min}=\ell_{m-1}$. First, we notice that $n_{\min}$ is of the form ℓ_p for some $p\geqslant 0$. This follows from theorem 1 which implies that $n_{\min}$ is the minimum of the values ℓ_p corresponding to the strata Σ_p such that $W(\boldsymbol{t}_0-\varepsilon(k))\in\Sigma_p$ for some value of k. Moreover, as $\partial_1^{\ell_m}\tau_W(\boldsymbol{t}_0)\neq 0$ and $P_2(\boldsymbol{\partial})=\partial_1^2/2$ it easily follows that $\partial_1^{\ell_m-2}\tau_W(\boldsymbol{t}_0-\varepsilon(k))\not\equiv 0$, so that $n_{\min}$ is of the form ℓ_p with $p<m$. Therefore $n_{\min}=\ell_{m-1}$.

Finally, from (20) we deduce

$$\partial_1^{\ell_{m-1}}\tau_W(\boldsymbol{t}_0-\varepsilon(k))=\frac{c}{k^m}+\mathcal{O}\left(\frac{1}{k^{m+1}}\right)$$

with $c\neq 0$, since otherwise $W(\boldsymbol{t}_0)$ would admit elements of degree $d<m$. The rest of the proof immediately follows. □

Expression (18) for $\hat{f}(k,\boldsymbol{t}_0)$ constitues a regularization of the Baker functions of $\mathrm{Gr}^{(2)}$ near the τ-function zeroes. A more general process of desingularizing the Baker functions of the whole Gr is provided by theorem 7.4 of [5].

4. A class of solutions of hKdV hierarchies

In view of the results of the above section we have that the known classes of τ-functions for the standard KdV hierarchy are to our disposal in order to generate solutions to the hKdV hierarchies. For example, we can take the class which characterizes the rational solutions (see [1–3, 15, 16]), vanishing as $x\longrightarrow\infty$. These τ-functions can be obtained by means of coordinate translations from the τ-functions τ_m associated with the subspaces W_m spanned by $\{k^s : s\in S_m\}$. Moreover, we can write these latter in the form

$$\tau_m(\boldsymbol{t})=\begin{vmatrix} h_m & h_{m+1} & \dots & h_{2m-1}\\ h_{m-2} & h_{m-1} & \dots & h_{2m-3}\\ h_{m-4} & h_{m-3} & \dots & h_{2m-5}\\ \dots & \dots & \dots & \dots\\ h_{2-m} & h_{3-m} & \dots & h_1 \end{vmatrix}\qquad m\geqslant 1 \tag{21}$$

where $h_i=h_i(\boldsymbol{t})$ are the Schur polynomials:

$$\exp\left(-\sum_{n\geqslant 0}t_{2n+1}k^{2n+1}\right)=1+\sum_{i\geqslant 1}h_i(\boldsymbol{t})k^i.$$

The first few of which are

$$\begin{aligned}&\tau_1=-t_1\qquad \tau_2=-\tfrac{1}{3}t_1^3+t_3\\ &\tau_3=\tfrac{1}{45}t_1^6-\tfrac{1}{3}t_1^3t_3+t_1t_5-t_3^2.\end{aligned}\tag{22}$$

Let us describe some solutions of the $\mathrm{hKdV}_{(2m+1)}$ hierarchies for $m=1,2$ which derive from these τ-functions.

Let us first consider τ_2. It can be factorized as

$$\tau_2=-\tfrac{1}{3}\prod_{i=0}^{2}(t_1-\epsilon^i\sqrt[3]{t_3})\qquad \epsilon:=\exp\frac{2\pi}{3}\mathrm{i}.$$

Thus, for $t_3\neq 0$ we have three patches $\boldsymbol{t}^{(i)}(\boldsymbol{t}_1)$ $(i=0,1,2)$ with associated functions

$$b_1^{(i)}=\epsilon^i\sqrt[3]{3t_3}.$$

Each of them determines a wavefunction of the hKdV$_{(3)}$ hierarchy. For example, for $i = 0$ we obtain

$$f = \exp\left(k\sqrt[3]{3x} + k^3 x + \sum_{n\geqslant 2} t_{2n+1}k^{2n+1}\right)\left(\frac{1}{k} - \frac{1}{k^2}\frac{1}{\sqrt[3]{3x}}\right) \qquad x := t_3$$

and the following solution of the hKdV$_{(3)}$ hierarchy

$$u_0 = \frac{4}{9x^2} \qquad u_1 = \frac{3}{(3x)^{4/3}} \qquad u_2 = \frac{2}{(3x)^{2/3}}.$$

The analysis of the solutions of the hKdV hierarchies provided by τ_3 is more involved. The discriminant of τ_3 with respect to t_1 is

$$\Delta(\boldsymbol{t}_1) = [(3t_3)^5 - (5t_5)^3]^2.$$

Hence, if $\Delta(\boldsymbol{t}_1) \neq 0$ the polynomial $\tau_3(t_1, \boldsymbol{t}_1)$, as a function of t_1, has simple roots only, so that we may define six patches $\boldsymbol{t}^{(i)}(\boldsymbol{t}_1)$ which lead to solutions of the hKdV$_{(3)}$ hierarchy. The corresponding functions $t_1 = b_1^{(i)}(\boldsymbol{t}_1)$ satisfy the constraint

$$t_3^2 + \frac{t_1^3}{3}t_3 - \frac{t_1^6}{45} - t_1 t_5 = 0 \tag{23}$$

which can be explicitly solved for the variable $x := t_3$ as

$$x = -\frac{t_1^3}{6} \pm \sqrt{\frac{t_1^6}{20} + t_1 t_5}.$$

Thus, one finds two real continuous branches $t_1 = b_1^{(a)}(x, t_5)$ $(a = 1, 2)$. Observe that the branch over the point $(x, t_1) = (\frac{(5t_5)^{3/5}}{3}, (5t_5)^{1/5})$, has a singular x-derivative at that point. Notice also that $b^{(2)}(x, t_5) = -b^{(1)}(-x, -t_5)$.

Let us consider now the case $\Delta(\boldsymbol{t}_1) = 0$; that is to say,

$$t_3 = \tfrac{1}{3}(5t_5)^{3/5}$$

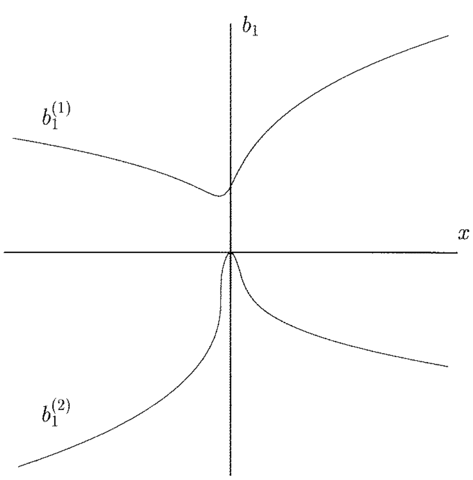

Figure 1. Implicit branches of equation (23) for $t_5 = -2$.

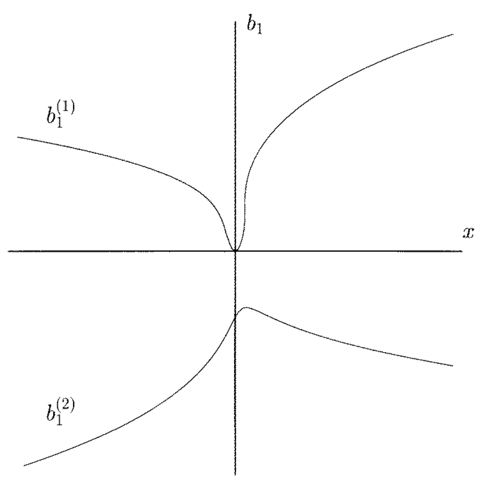

Figure 2. Implicit branches of equation (23) for $t_5 = 2$.

for a certain determination of the cubic root. Under this condition one finds

$$\tau_3 = \tfrac{1}{45}(t_1 - (5t_5)^{1/5})^3 \prod_{i=0}^{2}(t_1 - a_i(5t_5)^{1/5})$$

where a_i stand for the three different roots of $a^3 + 3a^2 + 6a + 5$. Thus, we obtain a patch $\boldsymbol{t}(\boldsymbol{t}_2)$ determined by

$$b_1 = (5x)^{1/5} \qquad b_2 = \tfrac{1}{3}(5x)^{3/5} \qquad x := t_5$$

which leads to a solution of the hKdV$_{(5)}$ hierarchy. Observe that $\tau_3(\boldsymbol{t}(\boldsymbol{t}_{(2)}) - \varepsilon(k)) \equiv 0$, and that the corresponding wavefunction is

$$f(k, \boldsymbol{t}_{(2)}) = \exp\left(k(5x)^{1/5} + k^3\frac{1}{3}(5x)^{3/5} + k^5 x + \cdots\right)\left(\frac{1}{k^2} - \frac{(5x)^{-\frac{1}{5}})}{k^3}\right).$$

The τ-functions of the KdV hierarchy of polynomial-type are relevant in the analysis of the motion of poles for the rational solutions of the KdV equation [16]

$$\partial_3 u = \partial_1^3 u - 6u\partial_1 u.$$

Suppose $\tau_W(t_1, \boldsymbol{t}_1)$ is one of these functions. From the results of [1] and [16] one may prove that for most values of $\boldsymbol{t}_1$ there exists a positive integer m such that τ_W can be factorized into ℓ_m different simple factors as

$$\tau_W(t_1, \boldsymbol{t}_1) = \prod_{i=1}^{\ell_m}(t_1 - p_i(\boldsymbol{t}_1))$$

so that the corresponding solution of the KdV hierarchy takes the form

$$u_W(t_1, \boldsymbol{t}_1) = \sum_{i=1}^{\ell_m} \frac{2}{(t_1 - p_i(\boldsymbol{t}_1))^2}.$$

It turns out that after substituting this expression into the KdV equation one finds [16]

$$\partial_3 p_i = 12\sum_{j\neq i}\frac{1}{(p_i - p_j)^2} \qquad \sum_{j\neq i}\frac{1}{(p_i - p_j)^3} = 0$$

and this constitutes a constrained flow of the Calogero–Moser hierarchy. Similar equations are obtained by using the higher members of the KdV hierarchy. On the other hand, according to the results of the present paper, each of the functions $p_i(\boldsymbol{t}_1)$ determines a solution of the hKdV$_{(3)}$ hierarchy associated with the patch

$$\boldsymbol{t}^{(i)}(\boldsymbol{t}_1) = (p_i(\boldsymbol{t}_1), \boldsymbol{t}_1).$$

Thus, from (3) the corresponding wavefunction is

$$\hat{f}^{(i)}(k, \boldsymbol{t}_1) = -\frac{\tau_W(\boldsymbol{t}^{(i)}(\boldsymbol{t}_1) - \varepsilon(k))}{\partial_1\tau_W(\boldsymbol{t}^{(i)}(\boldsymbol{t}_1))}.$$

For these solutions it readily follows that the equations of the hKdV$_{(3)}$ hierarchy reduce to partial differential equations for the ℓ_m functions $p_i(\boldsymbol{t}_1)$. They describe differential constraints for the *hypersurfaces* $t_1 = p_i(\boldsymbol{t}_1)$ in $\mathbb{C}^\infty$ involving several coordinates t_{2i+1}. For example, it is not hard to see that from the third equation of (16) one finds

$$\partial_3\left[\partial_5 p_i + (\partial_3 p_i)^2 + \partial_3\sum_{j\neq i}\frac{1}{p_i - p_j}\right] = 0.$$

In what concerns the higher hKdV hierarchies, they arise when manifolds of multiple zeros are present in the factorization of τ_W, so that they describe differential constraints for the *collisions* of manifolds of simple zeros.

Finally, we notice that the solutions of the hKdV hierarchies determined in this section involve in general implicit functions $b_i(\boldsymbol{t}_m)$. This type of solution also appears in the theory of the Harry Dym equation [17–19] which in turn is also described in the context of integrable hierarchies associated with generalized Schrödinger problems. Therefore, it may be expected that the analysis of this paper can be generalized to the integrable models characterized in [11].

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